100 Questions – Logic and Proofs

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Introduction. Learning to write proofs is an essential skill for mathematics, but one which requires a lot of practice. Every proof you write should aim to be *valid* (meaning mathematically correct) and *readable* (easy to follow and understand). You can learn to write good proofs by first reading well-written proofs, writing your own and getting feedback from others.

There are also many proof techniques, such as contrapositive, contradiction and mathematical induction. An understanding of logical expressions and arguments is useful, and covered in the worksheet.

We try to write questions that are focused on the proofs rather than other content, but some knowledge of set theory notation, elementary number theory and other general topics will be assumed. Prepare for some difficult questions!

LOGICAL SYMBOLS

Q1) If P and Q are logical propositions, we can construct the propositions $\neg P$, $P \vee Q$, $P \wedge Q$ and $P \to Q$. Write the meaning of each of these symbols and their truth tables.

Q2) If P and Q are true propositions and R is false, which of the following statements are true?

- **a)** $P \vee R$
- **b)** $(\neg P \vee Q) \wedge R$
- **c)** $(P \to R) \to \neg R$
- **d)** $(P \wedge \neg Q) \to (\neg P \vee Q)$

Q3) By constructing the truth tables or otherwise, justify the following:

- **a)** $\neg(P \vee Q) \equiv \neg P \wedge \neg Q$
- **b)** $\neg(P \wedge Q) \equiv \neg P \vee \neg Q$

These are known as de Morgan's laws.

Q4) By writing the truth tables, justify the logical equivalence

$$P \to Q \equiv \neg Q \to \neg P.$$

This is known as the *contrapositive*.

Q5) Justify $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$ and $P \wedge (Q \vee R) \equiv (P \wedge Q) \vee (P \wedge R)$.

Q6) Justify $(P \to Q) \wedge (Q \to R) \to (P \to R)$.

Q7) Under what conditions do we have $P \to Q$ but not $Q \to P$?

Q8) Write the following sentences as logical statements involving an implication $\to$.

a) If P is true, then Q is true.

b) P implies Q.

c) P is true whenever Q is true.

d) P is true only if Q is true.

e) P can't be true unless Q is true.

Q9) Explain the meaning of the *quantifiers* $\exists$ and $\forall$.

Q10) Determine whether the following statements are true, where m, n are integers.

a) $\exists n(n \text{ is prime})$

b) $\forall n(n \text{ is prime})$

c) $\exists m \exists n(m \leq n)$

d) $\exists m \forall n(m \leq n)$

e) $\forall m \exists n(m \leq n)$

f) $\forall m \forall n(m \leq n)$

Q11) If $P(n)$ is a logical proposition depending on n, which of the following is always true (regardless of the proposition P)?

a) $(\exists n(P(n))) \to (\forall n(P(n)))$

b) $(\forall n(P(n))) \to (\exists n(P(n)))$

Q12) Give an example of a proposition $P(n)$ where n is an integer such that $\neg \exists n(P(n))$ is true, but $\exists n(\neg P(n))$ is false. This shows that the supposed equivalence

$$\neg(\exists n(P(n))) \equiv \exists n(\neg P(n))$$

is **not** valid. It turns out that the correct equivalence is

$$\neg(\exists n(P(n))) \equiv \forall n(\neg P(n)).$$

Check that this works with your proposition P.

Q13) Write a statement without negation symbols which is equivalent to

$$\neg(\forall n \exists m(\neg P(m) \vee \neg Q(n)).$$

Q14) Suppose $\psi(n)$ is a proposition on integers such that

$$\psi(0) \wedge \forall n(\psi(n) \to (n = 0))$$

a) What does this statement say about ψ?

b) What is the proposition ψ?

Q15) Suppose $\phi(n)$ is a proposition on positive integers which satisfies the following logical statement:

$$\forall n \exists m((m \geq n) \wedge \phi(m)).$$

a) What does this statement say about ϕ?

b) How many positive integers satisfy $\phi(n)$?

Q16) Given a proposition $P(n)$, write a formal expression which translates as

"7 is the smallest positive integer for which P is true."

Q17) Convert the following statements about a positive integer n into logical statements.

a) n is an odd number.

b) n is divisible by 5.

c) n is not a power of two.

d) n is a composite number.

e) n is a prime number.

Q18) The following statement makes a claim about a sequence of integers $a_1, a_2, \ldots$. What claim is being made about the sequence?

$$\forall N \exists k \forall m((m \geq k) \rightarrow (a_m \geq N)).$$

(Hint: Replace N with a fixed number, like 1,000. What does it say now?)

Q19) We can make similar statements for real-valued function $f : \mathbb{R} \rightarrow \mathbb{R}$, known as ε–δ definitions. Let ε and δ represent positive real numbers. What specifically is being claimed about the function here?

$$\forall \varepsilon \exists \delta \forall x((0 < x < \delta) \rightarrow (0 < f(x) < \varepsilon)).$$

Q20) Translate the following into a logical statement:

"For any positive integer n, there is a prime number bigger than n."

If you like, you may let $\varrho(n)$ be the proposition "n is prime".

Q21) Suppose $\varrho(n)$ is the proposition "n is prime". Translate the following famous open problems about prime numbers into words. All variables are positive integers.

a) Legendre. $\forall n \exists p(\varrho(p) \wedge (n^2 \leq p \leq (n+1)^2))$

b) Fermat. $\forall n(n > 4) \rightarrow \neg\varrho(2^{2^n} + 1))$

c) Goldbach. $\forall n \exists p \exists q(\varrho(p) \wedge \varrho(q) \wedge ((n = 1) \vee (2n = p + q)))$

d) Feit–Thompson.
$\forall p \forall q((\varrho(p) \wedge \varrho(q) \wedge \exists k(k(p^q - 1)/(p - 1) = (q^p - 1)/(q - 1)) \rightarrow (p = q)))$.

EXAMPLES AND COUNTEREXAMPLES

When given a proposition of the form $(\exists n : P(n))$, you can prove it is true by finding an n for which $P(n)$ is true. When given a proposition of the form $(\forall n : P(n))$, you need to give a *general proof* of $P(n)$ which works for every n. However, if you are trying to disprove the proposition, you can give a single counterexample.

For the following statements, determine whether they are true or false, and provide a proper justification of why. These may involve examples, counterexamples and general proofs.

Q22) For all integers m and n, if $m \leq n$ then $m^2 \leq n^2$.

Q23) If the last digit of n is 5, then the last digit of n^2 is also 5.

Q24) Every square number is divisible by four.

Q25) If x and y are real numbers, then $xy = 0$ if and only if $x = 0$ or $y = 0$.

Q26) If $x_1 \neq y_1$ and $x_2 \neq y_2$, then $x_1 y_1 \neq x_2 y_2$.

Q27) If a leaves a remainder of 1 when dividing by three, then $5a$ leaves a remainder of 2.

Q28) A triangle has an *obtuse* angle if and only if it has two angles such that the sum of these angles is less than $90°$.

Q29) The average of a list of numbers can't be bigger than the maximum value in the list.

Q30) Real numbers a and b are both irrational if and only if $a + b$ is irrational.

Q31) $\forall m \exists n(n > 2^m)$

Q32) $\forall x \exists y \forall z(((x \leq z) \wedge (0 < x)) \rightarrow (0 < y < z))$

Q33) $\forall m \exists n \forall p(m + n = p)$

Q34) $\neg \exists x \forall y(y = 5x + 1)$

Q35) $\exists x \forall y \exists z (z = xy)$.

Q36) There is a unique solution to $x^2 = x$ in the real numbers.

Q37) An integer is divisible by 6 if and only if it divisible by 2 and divisible by 3.

Q38) The number 123123123 is prime.

Q39) There is a positive integer n such that the last digit of n^2 is 7. (Hint: Do you actually need to check *every* positive integer here?)

CONTRAPOSITIVE

Replacing an implication $P \to Q$ by $\neg Q \to \neg P$ can sometimes result in a statement which is easier to prove (or disprove). It is a common mistake to replace the implication with $Q \to P$. This is the *converse*, and is not equivalent.

Q40) Consider the statement

"If it is raining, the bus will be late."

- **a)** Write the contrapositive to the statement.
- **b)** Write the converse to the statement.
- **c)** What is a situation where the statement holds, but the converse doesn't?

Q41) Write the contrapositive to: "prime numbers have exactly two divisors".

Q42) Write the contrapositive to the statement: "students who spend more time studying are less likely to fail the course".

Q43) Write the contrapositive to: "reptiles are tetrapods with ectothermic metabolisms".

Q44) Prove by contrapositive that if a^2 is an odd integer, then so is a.

Q45) Prove by contrapositive that if $x \neq 0$ is an irrational number, then so is $1/x$.

Q46) Prove by contrapositive that if $x + y > 200$, then $x > 100$ or $y > 100$.

Q47) Prove that for $p > 2$, if p is prime then p is odd.

Q48) If n is a multiple of three, it can be written as the sum of three consecutive integers.

Q49) Call a real number x *Gausslike* if $x = a + b\sqrt{2}$ for integers a and b. Prove that if xy is not Gausslike, then at least one of x and y is not Gausslike.

Q50) In a court case, the police make the following argument:

> "The drug test we use has a 100% detection rate, which means if you have smoked cannabis recently you will always test positive. Since you tested positive, we can prove you recently smoked."

Is this argument valid? What should your lawyer say?

Q51) Your landlord tells you that if electricity prices rise or a new school is built in your neighbourhood, your rent will increase. Neither of those happened this year, so what can you expect about your rent?

MATHEMATICAL INDUCTION

Induction is a technique for proving a statement $P(n)$ for all positive integers $n \geq 1$. This is accomplished by proving the following two facts:

- $P(1)$,
- $P(k) \to P(k+1)$ for every $k \geq 1$.

The proposition $P(1)$ is called the *base case* and is often very easy to prove. The statement $P(k)$ is called the *inductive hypothesis*, and you allowed to use this proposition to prove $P(k+1)$.

There are also variations on induction, called *strong induction*. The most general case is to strengthen the inductive hypothesis to give

- $P(1)$,
- $(P(1) \wedge P(2) \wedge \cdots \wedge P(k)) \to P(k+1)$ for every $k \geq 1$.

Q52) Let $S(n) := 1 + 2 + \cdots + n$. We want to prove a simple formula which calculates $S(n)$ for every $n \geq 1$ using mathematical induction.

a) Find $S(5)$.

b) It is known that $S(499) = 124,750$. What is $S(500)$?

c) Assuming that $S(k) = \dfrac{k(k+1)}{2}$, find a similar formula for $S(k+1)$.

d) Prove that $S(n) = \dfrac{n(n+1)}{2}$ for every $n \geq 1$ by induction.

Q53) Guess a simple formula for $1 + 3 + \cdots + (2n-1)$ by computing the sum for $n = 1, 2, 3$. Prove this formula using induction.

Q54) Prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for every $n \geq 1$.

Q55) Prove that the last digit of every positive power of 6 is also 6.

Q56) Prove that $n^3 + 2n$ is divisible by 3 for every $n \geq 1$.

Q57) Prove that $7^{2n} - 1$ is divisible by 12 for every $n \geq 1$.

Q58) Prove that $8^n - 3^n$ is divisible by 5 for every $n \geq 1$.

Q59) Let $P(n)$ be the statement: $5^n + 1$ is divisible by 6.

- **a)** Show that $P(1)$ is true.
- **b)** Prove that $P(k) \to P(k+2)$.
- **c)** For which positive integers can you conclude the statement holds?
- **d)** Is the statement true for all positive integers?

Q60) Prove that $2^n \geq n + 1$ for every $n \geq 0$.

Q61) Prove that $\dfrac{(2n)!}{2^n}$ is an integer for every $n \geq 0$.

Q62) Find the flaw in the following strong induction proof, which 'proves' the proposition $P(n)$ that $2^n = 1$ for every $n \geq 0$.

- Proof of $P(0)$: We have $2^0 = 1$, since $a^0 = 1$ for every $a \neq 0$.
- Assumption: Assume that $2^j = 1$ for every $0 \leq j \leq k$.
- Proof of $P(k+1)$: We have

$$a^{k+1} = \frac{a^k \cdot a^k}{a^{k-1}} = \frac{1 \cdot 1}{1} = 1$$

using our assumptions.

Q63) Consider the following argument:

"1 is much less than a million. If k is much less than a million, then so is $k + 1$. Hence, every positive integer is much less than a million by mathematical induction."

How does this make you feel?

Q64) Suppose you have proven the statements

1. $P(2)$
2. $P(2) \wedge P(n) \to P(2n)$ for $n \geq 2$
3. $P(n) \to P(n-1)$ for $n \geq 2$

Is this sufficient to conclude that $P(n)$ is true for every integer $n \geq 1$?

Q65) A row of n books are placed on a shelf. If you can only swap adjacent books, how many swaps does it take to completely reverse the order of the books? Prove your formula by mathematical induction.

Q66) Let a_n be the sequence given by $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{2 + a_n}$ for every $n \geq 1$.

- **a)** Find the value of a_1, a_2 and a_3 using a calculator.
- **b)** Prove that $a_n < 2$ for every $n \geq 1$.

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Q67) Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{x}{1-x}.$$

Use induction to prove for every $n \geq 1$ that the n-fold function composition is given by

$$f^{(n)}(x) = f(f(f \cdots f(x))) = \frac{x}{1-nx}.$$

Q68) The *Fibonacci numbers* is the sequence $0, 1, 1, 2, 3, 5, 8, \ldots$ is is defined by $F_0 = 0$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for every $n \geq 1$. These numbers naturally occur as the number of petals on a sunflower, for example. Let $\Psi(n)$ be the proposition

$$F_0 + F_1 + F_2 + \cdots + F_{n-1} = F_{n+1} - 1$$

for every $n \geq 1$, which we will prove by a variant of induction.

- **a)** Check whether $\Psi(4)$ is true.
- **b)** As base cases for the induction, prove $\Psi(1)$ and $\Psi(2)$.
- **c)** By assuming $\Psi(k)$, prove $\Psi(k+2)$. Why do we need two base cases?

Q69) The Fibonacci numbers have a closed form involving the *golden ratio*. We will prove a related result here. The golden ratio is the irrational value $\varphi = (1 + \sqrt{5})/2$. We want to prove the proposition $\Phi(n)$ that $F_n \leq \varphi^n$ for every $n \geq 0$.

- **a)** Prove that $\varphi^2 = 1 + \varphi$.
- **b)** Prove the base cases $\Phi(0)$ and $\Phi(1)$.
- **c)** Prove that $(\Phi(k-1) \wedge \Phi(k)) \to \Phi(k+1)$.

Q70) Prove that every integer $n \geq 2$ can be expressed as a product of prime numbers.

Q71) Consider a grid of size $2^n \times 2^n$ with a single square missing. Prove by induction that you can cover the remaining grid by -shaped tiles.

PROOF BY CONTRADICTION

In a proof by contradiction, we assume that the proposition P we are trying to prove is **false**, and derive a mathematical contradiction, such as the fact that $1 = 0$, or that P is in fact true. We can draw one of two conclusions from this:

- Mathematics has a fundamental error which you just found.
- Your assumption that P was false is incorrect.

We conclude that the second case is more reasonable, so P is true.

Q72) Prove that there are no integer solutions to $24a + 6b = 1$. (Hint: Assume a solution exists, and prove that a or b are not both integers.)

Q73) Prove that if x is irrational and y is rational, then the sum $x + y$ is always irrational.

Q74) Prove that there is no triangle whose side lengths are all integers, and whose base side lengths are both odd. (Hint: Does the hypotenuse have even or odd length? Use Pythagoras.)

Q75) We recreate Euclid's beautiful proof that there are infinitely many primes.

a) Given integers $p_1, p_2, \ldots, p_n$ with each $p_i \geq 2$, explain why $p_1 p_2 \cdots p_n + 1$ is not divisible by any of the original numbers.

b) Prove that there are infinitely many primes. (Hint: Assume there are only finitely many primes, and that $p_1, \ldots, p_n$ is a complete list.)

Q76) Prove that $\log_2(3)$ is irrational. (Hint: Assume it is rational).

Q77) Pick apart the following supposed proof by contradiction, whose claim is that every number bigger than 8 is prime.

"Assume that there are no primes bigger than 8. But 11 is bigger than 8, and the only divisors of 11 are 1 and itself. Hence, we obtain a contradiction, so the claim is correct."

Q78) Consider the following proof by contradiction, which proves that the sum of two even numbers is always even.

"Assume (for a contradiction) that n and m are even numbers whose sum is odd. Since they are even, $n = 2i$ and $m = 2j$ for some integers i and j, so $n + m = 2(i + j)$ is an even number. This is not odd, which gives a contradiction."

This proof is mathematically valid, but it is not a good proof. Why?

Q79) Suppose P, Q R and S are propositions satisfying the following:

1. $(\neg Q \vee R) \rightarrow P$
2. $Q \rightarrow S$
3. $R \vee \neg S$

Prove P by contradiction.

HARDER QUESTIONS

Q80) Prove or disprove that if $2^n - 1$ is composite, then n is composite.

Q81) Prove that the equations

$$Ax + By = U \quad \text{and} \quad Cx + Dy = V$$

have a unique solution if and only if $AD - BC \neq 0$.

Q82) Prove that every square number leaves a remainder of 0 or 1 when dividing by 4.

Q83) Suppose x, y are positive real numbers with $x \neq y$. Prove that

$$\frac{x}{y} + \frac{y}{x} > 2.$$

Q84) A function $f : X \rightarrow Y$ is *surjective* if every y in Y has an x in X such that $f(x) = y$.

- **a)** Write the statement "f is surjective" using logical symbols.
- **b)** Prove that if $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are surjective functions, then the composition $g \circ f : X \rightarrow Z$ is also surjective.
- **c)** Prove that the converse is not true. That is, find functions such that $g \circ f$ is surjective, but f and g are not both surjective.

Q85) Suppose we have 100 positive integers $n_1, n_2, \ldots, n_{100}$ such that

$$\frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_{100}} = 1.$$

Prove that at least one of the integers is even.

Q86) Suppose $P(x)$ is a proposition on an infinite set X which satisfies

$$\forall x \forall y ((P(x) \wedge P(y)) \rightarrow (x = y)).$$

For how many x can $P(x)$ be true?

Q87) Suppose $\varrho(n)$ is the proposition "n is prime". There is a famous result of Yitang Zhang proved in 2013 which claims the following:

$$\exists N(N \leq 70,000,000 \land \forall n \exists p(p \geq n \land \varrho(p) \land \varrho(p+N))).$$

What claim is being made? Try to express this as simply as possible.

Q88) Prove that there do not exist integers m and n such that $2n^2 = m^2$.

Q89) Give a proof by contradiction that $\sqrt{2}$ is an irrational number. (Hint: Use the previous question!)

Q90) Here is a *nonconstructive* existence proof, where we prove an object exists without ever specifically finding it. Our statement will be: there exist irrational numbers a and b such that a^b is rational.

a) Prove the statement under the assumption that $r = \sqrt{2}^{\sqrt{2}}$ is rational.

b) Prove the statement under the assumption that $r = \sqrt{2}^{\sqrt{2}}$ is irrational by considering $r^{\sqrt{2}}$.

Q91) [Assumes some calculus] A function $f : \mathbb{R} \to \mathbb{R}$ is called *weakly increasing* if

$$\forall x \forall y((x \leq y) \to (f(x) \leq f(y)).$$

Prove that a differentiable function is weakly increasing if and only if $f'(x) \geq 0$ for every x.

Q92) Consider a line of 9 people. A *descent* is a pair of adjacent people where the shorter person is on the right. An *inversion* is a pair of people (not necessarily adjacent) with the shorter person on the right.

a) Prove there are no inversions if and only if there are no descents.

b) Is there exactly one inversion if and only if there is exactly one descent? Prove your claim.

Q93) Suppose X is a set with a binary operation $\oplus$, meaning for every x and y in X, we are given an element $x \oplus y$ in X. Also suppose ι is a specific element such that $x \oplus \iota = x$ and $\iota \oplus x = x$ for every x. Prove that only one element in X can satisfy this property.

Q94) What's the largest integer n which can't be written as $n = a + b$, where $a, b \geq 2$ are composite numbers? Prove your claim. (Hint: Consider $a = 4, 6, 8$.)

Q95) The *probabilistic method* is another technique for nonconstructive proofs. Many of the proofs are quite advanced, but here is a silly one. We prove the following: if you paint 90% of a sphere red and paint the remaining 10% blue, you can always can fit a cube inside the sphere whose corners are all red.

a) Place a cube randomly inside the sphere, with the vertices lying on the sphere. What is the probability that a given vertex of the cube will be on the red part of the sphere?

b) Find the expected number of red corners.

c) Hence, explain why *some* cube has all red corners.

Q96) Prove the AM–GM inequality: for positive real numbers $x_1, \ldots, x_n$ for $n \geq 1$:

$$\frac{x_1 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 \cdots x_n}.$$

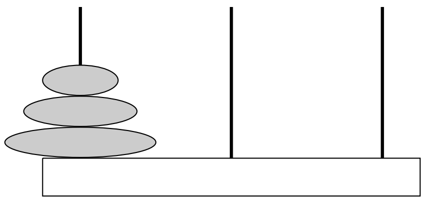
(Hint: Consider the implications from Q64).)

Q97) A foreign currency has bank notes of only \$4 and \$5. Prove that you can pay any dollar amount of at least \$12 with only these types of notes.

Q98) By drawing 3 chords through a circle, you can split the circle into a maximum of 7 regions. For $0 \leq n \leq 4$, find the maximum number of regions which the circle can be split into by drawing n chords. Find a general formula and prove it by induction.

Q99) Recall the Fibonacci numbers from Q68). Prove that every positive integer can be written as a sum of distinct Fibonacci numbers.

Q100) In this question we find an optimal solution for the Towers of Hanoi puzzle. Consider three poles, the first of which has n stones, stacked from largest to smallest. We want to move these stones to the third pole. We can only move one disc at a time, and we can never have a larger stone on top of a smaller one.



Find the minimum number of moves needed to transfer all stones onto the third pole when there are n stones for $n \leq 4$. Guess a formula, and give a proof by induction that your solution is optimal.